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A NON-REFLEXIVE GROTHENDIECK SPACE THAT DOES NOT CONTAIN l_∞

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ABSTRACT

A compact space S is constructed such that, in the dual Banach space $\mathscr{C}(S)^*$, every weak* convergent sequence is weakly convergent, while $\mathscr{C}(S)$ does not have a subspace isomorphic to l_∞. The construction introduces a weak version of completeness for Boolean algebras, here called the Subsequential Completeness Property. A related construction leads to a counterexample to a conjecture about holomorphic functions on Banach spaces. A compact space T is constructed such that $\mathscr{C}(T)$ does not contain l_∞ but does have a "bounding" subset that is not relatively compact. The first of the examples was presented at the International Conference on Banach spaces, Kent, Ohio, 1979.

1. A Grothendieck space

A Banach space X is called a Grothendieck space if every weak* convergent sequence in the dual space X^* is also weakly convergent. The best known examples of non-reflexive Grothendieck spaces are the spaces $\mathscr{C}(S)$, where S is compact and extremally disconnected, or, more generally, where S is an F-space [9]. It is known that such spaces $\mathscr{C}(S)$ contain isometric copies of l_∞ (subject to the Continuum Hypothesis, in the case of F-spaces), and the question has been raised (for instance, on page 180 of [1]) of whether every non-reflexive Grothendieck space has a subspace isomorphic to l_∞. The construction given here shows that this is not the case. Since the space constructed is of the type $\mathscr{C}(S)$, with S compact, it also answers negatively a question posed by Pełczyński (see also page 201 of [5]), whether every $\mathscr{C}(S)$ space necessarily contains either l_∞ or a complemented c_0. A different counterexample to these conjectures has been found independently by M. Talagrand [11]. His construction depends on the Continuum Hypothesis, but the Grothendieck space $\mathscr{C}(T)$ which he obtains has the stronger property that it does not have l_∞ as a quotient (or, equivalently, thanks to CH, that it does not contain $l_1(\omega_1)$ as a subspace).

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The space S which we construct here will be obtained as the Stone space of a certain Boolean subalgebra $\mathfrak{A}$ of $\mathcal{P}\omega$, the algebra of all subsets of the natural numbers. An account of the relationship between Boolean algebras and their Stone spaces may be found in §16 of [10], for example. However, it may be worth remarking that, once our subalgebra $\mathfrak{A}$ of $\mathcal{P}\omega$ has been constructed, the Banach space we are interested in has a concrete realization, since $\mathscr{C}(S)$ can be identified with the closed linear span in l_∞ of the indicator functions of the elements of $\mathfrak{A}$. When we are thinking of it in this way, we shall denote this space by $X_{\mathfrak{A}}$. We now start by introducing a piece of terminology for what will turn out to be the crucial property of our algebra.

1A Definition. We say that a Boolean algebra $\mathfrak{A}$ has the *Subsequential Completeness Property* if, whenever $(A_n)_{n\in\omega}$ is a disjoint sequence in $\mathfrak{A}$, there is an infinite subset M of ω such that $(A_m)_{m\in M}$ has a least upper bound in $\mathfrak{A}$.

In the case where $\mathfrak{A} = \mathfrak{A}(S)$, the algebra of all open and closed subsets of a totally disconnected compact space S (equivalent to saying that S is the Stone space of $\mathfrak{A}$), the above property means that whenever (A_n) is a disjoint sequence of open and closed sets, there is an infinite subset M of ω such that the closure of $\bigcup_{m\in M} A_m$ is open in S.

1B Proposition. *Let S be a totally disconnected compact space. If the algebra $\mathfrak{A}(S)$ has the SCP then $\mathscr{C}(S)$ is a Grothendieck space.*

Proof. Let (μ_n) be a weak* null sequence in $\mathscr{C}(S)^*$. We have to show that $R = \{\mu_n : n \in \omega\}$ is weakly relatively compact, and to do this it will suffice to prove that if $(A_m)_{m\in\omega}$ is a disjoint sequence in $\mathfrak{A}(S)$ then $\mu(A_m)\to 0$ uniformly over $\mu \in R$. (This version of the criterion for weak compactness may be found, for instance, as Theorem 83C of [3].)

So let us suppose that (A_m) is a disjoint sequence in $\mathfrak{A}(S)$ and that $\mu_n(A_n) \geqq \delta > 0$ for all n. We may also suppose, by Rosenthal's Lemma ([5], or page 18 of [1]), that for all n

$$|\mu_n|(\bigcup\{A_m : n \neq m \in \omega\}) < \tfrac{1}{2}\delta.$$

Now let $(\sigma(\alpha))_{\alpha\in\omega_1}$ be an uncountable almost disjoint family of subsets of ω. (Recall that "almost disjoint" means that $\sigma(\alpha)\cap\sigma(\beta)$ is finite whenever $\alpha \neq \beta$.) For each α, choose an infinite subset $\tau(\alpha)$ of $\sigma(\alpha)$ such that the closure of $\bigcup_{n\in\tau(\alpha)} A_n$ is open; call these closures B_α. Then the sets $C_\alpha = B_\alpha \setminus \bigcup_{n\in\tau(\alpha)} A_n$ are pairwise disjoint, and so there exists an α such that $|\mu_n|(C_\alpha) = 0$ for all $n \in \omega$. We now see that for all $n \in \tau(\alpha)$

$$\langle \mu_n, 1_{B_\alpha} \rangle = \mu_n\Big(C_\alpha \cup \bigcup_{n \in \tau(\alpha)} A_n \Big) > \tfrac{1}{2}\delta,$$

contradicting the assumption that $\mu_n \to 0$ in the weak* topology.

REMARK. The Grothendieck Property for Boolean algebras has been considered by a number of authors, who have also looked at other measure-theoretical properties in this context. A survey of this material can be found in [7]. With a little more work, one can show that a Boolean algebra with the SCP also enjoys the so-called "Vitali–Hahn–Saks Property".

1C PROPOSITION. *Let S be compact and suppose that l_∞ embeds isomorphically in $\mathscr{C}(S)$. Then there is an infinite subset N of S such that $\bar{L} \cap \bar{M} = \emptyset$ whenever L, M are disjoint subsets of N. If T is a dense subset of S we may choose N to be contained in T.*

PROOF. The proof is almost entirely based on ideas from Rosenthal's paper [6]. Let $u : l_\infty \to \mathscr{C}(S)$ be an embedding and assume that for all $\xi \in l_\infty$ we have $\|\xi\| \leqq \|u\xi\| \leqq K \|\xi\|$.

For each $n \in \omega$ choose $s_n \in T$ such that $|(ue_n)(s_n)| \geqq 1$, e_n denoting the usual unit vector in l_∞. Consider the elements $\nu_n = u^*(\delta(s_n))$ of $(l_\infty)^*$ as finitely additive measures on ω. By an application of Rosenthal's Lemma, there exists an infinite subset D of ω such that, for all $n \in D$, $|\nu_n|(D \backslash \{n\}) < \frac{1}{3}$. Put $N = \{s_n : n \in D\}$ and let $M = \{s_n : n \in C\}$ be any subset of N. Consider $f_C = u(1_C) \in \mathscr{C}(S)$. We have

$$f_C(s_n) = \begin{cases} (ue_n)(s_n) + \nu_n(C_n \backslash \{n\}) & \text{if } n \in C, \\ \nu_n(C) & \text{if } n \notin C. \end{cases}$$

Thus, $|f_C(s_n)| > \frac{2}{3}$ if $n \in C$ and $|f_C(s_n)| < \frac{1}{3}$ if $n \notin C$. This assures us that $\bar{M} \cap \overline{(N \backslash M)} = \emptyset$.

REMARK. The above Proposition shows that a necessary condition for l_∞ to embed in $\mathscr{C}(S)$ is that S should have a subset homeomorphic to $\beta\omega$, the Stone–Čech compactification of the natural numbers. For if N is the subset of S constructed above, all indicator functions 1_M $(M \subseteq N)$ extend to continuous functions on S, and hence $\bar{N}$ is homeomorphic to $\beta\omega$. That this condition is not sufficient may be seen by taking $S = \{-1, 1\}^{2^\omega}$ and applying Hagler's results on subspaces of $\mathscr{C}(S)$ where S is dyadic [4].

We now come to the construction of our Boolean algebra, which will be achieved by transfinite induction, employing the following lemma. The author is grateful to the referee for suggesting a simplified proof.

1D Lemma. *Let $\gamma < 2^\omega$ be an ordinal and $\mathfrak{A}$ be a Boolean subalgebra of $\mathcal{P}\omega$ with $|\mathfrak{A}| \leqq |\gamma|$. Assume further that there is a family $(M_\beta, N_\beta)_{\beta<\gamma}$ of pairs of subsets of ω, with $M_\beta \subset N_\beta$ for all β, such that*

$$(1) \qquad M_\beta \neq N_\beta \cap A \qquad \text{for all } A \in \mathfrak{A}, \quad \beta < \gamma.$$

Then, given any disjoint sequence $(A_n)_{n\in\omega}$ in $\mathfrak{A}$, there is an infinite subset σ of ω such that $M_\beta \neq N_\beta \cap A$ for all $\beta < \gamma$ and all A in the algebra $\mathfrak{A}_\sigma$ generated by $\mathfrak{A}$ and $A_\sigma = \bigcup_{n\in\sigma} A_n$.

Proof. Note first that, for each set B, the elements of the algebra generated by $\mathfrak{A}$ and B have the form $A \cup (A' \cap B) \cup (A'' \setminus B)$, where A, A', A'' are disjoint elements of $\mathfrak{A}$.

Let Σ be a collection of 2^ω almost disjoint infinite subsets of ω and assume, if possible, that no A_σ ($\sigma \in \Sigma$) satisfies the conclusion of the lemma. Then for each $\sigma \in \Sigma$ there exist disjoint $A, A', A'' \in \mathfrak{A}$ and $\beta < \gamma$ such that

$$N_\beta \cap (A \cup (A' \cap A_\sigma) \cup (A'' \setminus A_\sigma)) = M_\beta.$$

Since there are fewer than 2^ω choices for (A, A', A'', β) there must exist distinct $\sigma, \tau \in \Sigma$ for which the same choice may be made. In particular, we shall have

$$(2) \qquad N_\beta \cap (A' \cap A_\sigma) = M_\beta \cap A' = N_\beta \cap (A' \cap A_\tau),$$

$$(3) \qquad N_\beta \cap (A'' \setminus A_\sigma) = M_\beta \cap A'' = N_\beta \cap (A'' \setminus A_\tau).$$

It follows from (2) that

$$M_\beta \cap A' = N_\beta \cap (A' \cap A_\sigma \cap A_\tau).$$

Since $\sigma \cap \tau$ is finite, the intersection $A_\sigma \cap A_\tau = A_{\sigma\cap\tau}$ is in $\mathfrak{A}$, and so also is $A' \cap A_{\sigma\cap\tau}$. Similarly, it follows from (3) that

$$M_\beta \cap A'' = N_\beta \cap (A'' \cap A_{\sigma\cap\tau}).$$

Finally, we observe that $M_\beta = N_\beta \cap B$, where $B = A \cup (A' \cap A_{\sigma\cap\tau}) \cup (A'' \setminus A_{\sigma\cap\tau}) \in \mathfrak{A}$, contradicting (1).

1E Proposition. *There is an algebra $\mathfrak{A}$ of subsets of ω, containing the finite subsets, and having the Subsequential Completeness Property, but such that for no infinite $N \subset \omega$ do we have $\mathcal{P}N = \{N \cap A : A \in \mathfrak{A}\}$.*

Proof. We construct $\mathfrak{A}$ by transfinite induction as the union of an increasing family $\mathfrak{A}_\alpha$ ($\alpha \in 2^\omega$) of subalgebras of $\mathcal{P}\omega$. We start by taking $\mathfrak{A}_0$ to consist of all finite and all cofinite subsets of ω. We enumerate the disjoint sequences in $\mathfrak{A}_0$ as

$$(A_n(0,\zeta))_{n\in\omega} \qquad (\zeta \in 2^\omega).$$

We also enumerate the infinite subsets of ω as N_α ($\alpha \in 2^\omega$). We choose a subset M_0 of N_0 which is not of the form $N_0 \cap A$ with $A \in \mathfrak{A}_0$. (In this first case we merely ensure that M_0 and $N_0 \setminus M_0$ are both infinite.) Before proceeding with the rest of the construction, we fix a surjection $\rho : 2^\omega \to 2^\omega \times 2^\omega$ having the property that if $\rho(\xi) = (\eta, \zeta)$ then $\eta < \xi$.

Now suppose that for each $\alpha < \gamma$ we have obtained a subalgebra $\mathfrak{A}_\alpha$ of $\mathcal{P}\omega$ with $|\mathfrak{A}_\alpha| = \max\{\omega, |\alpha|\}$, an infinite subset M_α of N_α and an enumeration $(A_n(\alpha,\xi))_{n\in\omega}$ ($\xi \in 2^\omega$) of the disjoint sequences in $\mathfrak{A}_\alpha$. Assume also that $\mathfrak{A}_\alpha \subset \mathfrak{A}_\beta$ for all $\alpha < \beta < \gamma$, and that for all $\alpha, \beta < \gamma$ we have $M_\beta \not\in \{N_\beta \cap A : A \in \mathfrak{A}_\alpha\}$.

Let us write (η, ζ) for $\rho(\gamma)$ and apply 1D with $A_n = A_n(\eta,\zeta)$, $\mathfrak{A} = \bigcup_{\beta<\gamma} \mathfrak{A}_\beta$. We obtain an infinite subset σ of ω such that, if $\mathfrak{A}_\gamma$ is the algebra generated by $\mathfrak{A}$ and $\bigcup_{n\in\sigma} A_n$, we have

$$M_\beta \neq N_\beta \cap A \qquad \text{for all } A \in \mathfrak{A}_\gamma \text{ and all } \beta < \gamma.$$

We fix an enumeration $(A_n(\gamma,\xi))$ ($\xi \in 2^\omega$) of the disjoint sequences in $\mathfrak{A}_\gamma$ and choose a subset M_γ of N_γ that is not of the form $N_\gamma \cap A$ with $A \in \mathfrak{A}_\gamma$. Such a choice is possible, since by hypothesis and construction we have $|\mathfrak{A}_\gamma| = \max\{\omega, |\gamma|\} < 2^\omega$.

Finally, we put $\mathfrak{A} = \bigcup\{\mathfrak{A}_\alpha : \alpha \in 2^\omega\}$ and have to check two properties of $\mathfrak{A}$. Firstly, note that if N is an infinite subset of ω then $N = N_\gamma$ for some $\gamma \in 2^\omega$ and that by construction there is no $A \in \mathfrak{A}$ with $N_\gamma \cap A = M_\gamma$. Now let (A_n) be a disjoint sequence in $\mathfrak{A}$. Since 2^ω is not cofinal with ω, there exists $\alpha \in 2^\omega$ such that each A_n is in $\mathfrak{A}_\alpha$, and so $(A_n) = (A_n(\alpha,\xi))$ for some $\xi \in 2^\omega$. If γ is an ordinal with $\rho(\gamma) = (\alpha, \xi)$ then there is an infinite subset σ of ω such that $\bigcup_{n\in\sigma} A_n \in \mathfrak{A}_\gamma$. Thus $\mathfrak{A}$ has the Subsequential Completeness Property.

1F THEOREM. *There is an infinite compact space S such that $\mathcal{C}(S)$ is a Grothendieck space with no subspace isomorphic to l_∞.*

PROOF. We take S to be the Stone space of the algebra $\mathfrak{A}$ constructed in Proposition 1E. So S is totally disconnected and $\mathfrak{A}(S)$ can be identified with $\mathfrak{A}$; $\mathcal{C}(S)$ is thus a Grothendieck space by Proposition 1B. Also, since $\mathfrak{A} \subset \mathcal{P}\omega$ and $\{n\}$ is in $\mathfrak{A}$ for all $n \in \omega$, we see that ω can be identified with a dense open subset of S. Now if B, C are subsets of ω and the closures $\bar{B}, \bar{C}$, taken in S, are disjoint, there exists an open and closed $U \subseteq S$ with $\bar{B} \subseteq U$, $U \cap \bar{C} = \emptyset$. Hence, there exists $A \in \mathfrak{A}$ with $B \subseteq A$, $A \cap C = \emptyset$. Thus, by the properties of $\mathfrak{A}$, there does not exist an infinite subset N of ω such that $\bar{M}$ and $\overline{(N - M)}$ are disjoint for all $M \subset N$. So l_∞ does not embed in $\mathcal{C}(S)$ by Proposition 1C.

As mentioned before, the Continuum Hypothesis example given by Talagrand [11] has the stronger property of not containing $l_1(\omega_1)$. I am grateful to Spiros Argyros, who pointed out that this definitely is not the case for the example given here.

Recall that a family $(A_\alpha)_{\alpha\in\tau}$ in a Boolean algebra is said to be *independent* if for every pair of disjoint finite subsets F, G of τ the intersection $\bigcap_{\alpha\in F} A_\alpha \cap \bigcap_{\beta\in G} A_\beta^c$ is non-trivial. (A_β^c denotes the complement of A_β.) If S is the Stone space of $\mathfrak{A}$, then $\mathfrak{A}$ has an independent family of cardinality κ if and only if S allows a continuous surjection onto $\{-1,1\}^\kappa$, which is if and only if $l_1(\kappa)$ embeds isometrically in $\mathscr{C}(S)$.

1G Proposition (Argyros). *If $\mathfrak{A}$ is an infinite Boolean algebra with the Subsequential Completeness Property then $\mathfrak{A}$ contains an uncountable independent family.*

Proof. It is easy to see that we can find in $\mathfrak{A}$ a disjoint sequence $(A_n)_{n\in\omega}$ such that each A_n contains the members of an independent sequence $(A_{n,j})$. The independent family we construct will consist of elements of the form $\bigvee_{n\in M} A_{n,\phi(n)}$ for suitable infinite subsets M of ω and maps $\phi : M \to \omega$. The construction of a family of pairs $(M_\alpha, \phi_\alpha)_{\alpha\in\omega_1}$ proceeds by transfinite induction.

Suppose that γ is a countable ordinal and that M_α, ϕ_α $(\alpha<\gamma)$ have been obtained already. Assume also that, for all $\alpha<\gamma$, the supremum $B_\alpha = \bigvee_{n\in M_\alpha} A_{n,\phi_\alpha(n)}$ exists in $\mathfrak{A}$, and that, for all $\alpha<\beta<\gamma$, the sets $M_\beta \setminus M_\alpha$ and $\{n\in M_\beta : \phi_\beta(n) \leqq \phi_\alpha(n)\}$ are both finite. First we choose an infinite $N\subset\omega$ such that $N\setminus M_\alpha$ is finite for all $\alpha<\gamma$, and then, by a standard diagonalization argument, a function $\psi : N\to\omega$ such that $\psi(n)$ is eventually greater than each $\phi_\alpha(n)$. We now use the SCP to obtain an infinite $M_\gamma \subset N$ such that $\bigvee_{n\in M_\gamma} A_{n,\psi(n)}$ exists in $\mathfrak{A}$, and set $\phi_\gamma = \psi|_M$.

We now show that $(B_\alpha)_{\alpha\in\omega_1}$ is independent. Let F, G be disjoint finite subsets of ω_1 and choose $n\in\omega$ such that n is in each M_α $(\alpha\in F\cup G)$ and such that, moreover, the integers $\phi_\alpha(n)$ $(\alpha\in F\cup G)$ are distinct. For each α we have $A_n \cap B_\alpha = A_{n,\phi_\alpha(n)}$, so that the intersection $\bigcap_{\alpha\in F} B_\alpha \cap \bigcap_{\beta\in G} B_\beta^c$ is non-empty, by the independence of $(A_{n,j})$.

2. Bounding subsets

Some of the ideas used in the construction just given can be employed to give a counterexample to a conjecture of Dineen and Schottenloher about holomorphic functions on Banach spaces. If X is a (complex) Banach space and B is a subset of X we say that B is *bounding* (in X) if every holomorphic function $f: X \to \mathbf{C}$ is bounded on B. In any weakly compactly generated Banach space the bounding subsets are exactly the relatively compact sets, but Dineen showed in [2] that the unit vectors e_n form a bounding subset of l_∞. It was conjectured [8] that a Banach space contains a bounding subset which is not relatively compact if and only if it has a subspace isomorphic to l_∞.

To give a counterexample to this, we introduce a piece of ad hoc terminology for a property of certain subspaces of l_∞.

2A DEFINITION. Let Λ denote the set of all pairs (K, λ) where K is an infinite subset of ω and

$$\lambda : \mathcal{P}K \to l_\infty; \qquad F \mapsto \lambda^F$$

is a function satisfying

(i) $|\lambda_n^F| = \begin{cases} 0 & (n \notin F), \\ 1 & (n \in F), \end{cases}$

(ii) $\lambda_n^F = \lambda_n^G$ whenever $F \cap [0, n] = G \cap [0, n]$.

We shall say that a subspace X of l_∞ has *Property* (Λ) if X contains the unit vectors $e_n = (0, \cdots, 0, 1, 0, \cdots)$ and if, for every $(K, \lambda) \in \Lambda$, there is an infinite subset J of K with $\lambda^J \in X$.

2B PROPOSITION. *If X is a subspace of l_∞ and has Property (Λ) then the unit vectors e_n form a bounding subset of X.*

PROOF. We shall be content to indicate the way in which Dineen's proof from [2] really does establish this result. If the unit vectors do not form a bounding subset then, without loss of generality, there exists a sequence of integers $n_1 < n_2 < \cdots$ and bounded symmetric n_j-linear forms A_{n_j} on X such that

$$\hat{A}_{n_j}(e_j) = A_{n_j}(e_j, \cdots, e_j) = 1,$$

while, for all $x \in X$,

$$|\hat{A}_{n_j}(x)|^{1/n_j} \to 0 \qquad \text{as } j \to \infty.$$

Dineen showed how to obtain an infinite subset $K = \{k(1), k(2), \cdots\}$ of ω having the property that, for all l,

$$\sup_{y \in \text{ball } l_\infty(C_l)} \sum_r \binom{n_{k(l)}}{r} \| A_{n_{k(l)}}(y)^{n_{k(l)}-r} \|_{C_l^\perp} \leqq 1/n_{k(l)}!.$$

In the above expression, $C_l = \{k(1), \cdots, k(l)\}$, $C_l^\perp = K \setminus C_l$, and, for an n-linear form A, $\|A(y)^{n-r}\|_{C_l^\perp}$ denotes the norm of the r-linear form

$$z \mapsto A(y, \cdots, y, z, \cdots, z) \qquad \text{on } X \cap l_\infty(C_l^\perp).$$

The remainder of Dineen's proof involves obtaining, by an inductive process, complex numbers λ_k with $|\lambda_k| = 1$ $(k \in K)$, in such a way that, if we write

$$\lambda_j^l = \begin{cases} \lambda_j & \text{if } j \in K \cap [0, k(l)], \\ 0 & \text{if not,} \end{cases}$$

we have $\hat{A}_{n_{k(l)}}(\lambda^l) \geqq 1$ for all l.

The complicated inequality above assures us that

$$\hat{A}_{n_{k(l)}}(z) \geqq 1 - 1/n_{k(l)}!$$

whenever z ball$(X \cap l_\infty(K))$ and $z_j = \lambda_j$ for $j \in K \cap [0, k(l)]$. Thus, if we knew that the sequence λ^K given by

$$\lambda_j^K = \begin{cases} \lambda_j & (j \in K) \\ 0 & (j \notin K) \end{cases}$$

was in our space X, we should have a contradiction, since $\limsup_{j\to\infty} \hat{A}_{n_j}(\lambda^K)$ would be at least 1.

Of course, λ^K need not be in X, but the property of K which is used is inherited by all its infinite subsets and so the construction of the λ's could have equally well been carried out starting with any infinite $J \subset K$. The inductive process is such that the choice at the kth stage depends only on $J \cap [0, k]$, so that we have, in fact, got an element (K, λ) of Λ. Property (Λ) now tells us that X contains some λ^J, which is what we want.

2C Proposition. *There is an algebra $\mathfrak{A}$ of subsets of ω, such that the closed linear span $X_{\mathfrak{A}}$ of $\{1_A : A \in \mathfrak{A}\}$ in l_∞ has Property (Λ), but such that for no infinite $N \subset \omega$ do we have $\mathfrak{S}N = \{N \cap A : A \in \mathfrak{A}\}$.*

Proof. Enumerate the elements of Λ as $(K_\alpha, \lambda_\alpha)$ $(\alpha \in 2^\omega)$ and the infinite subsets of ω as N_α $(\alpha \in 2^\omega)$. Let $\mathfrak{A}_0$ consist of all finite and all cofinite subsets of ω and perform an inductive construction.

Suppose that for each $\alpha < \gamma$ we have obtained a subalgebra $\mathfrak{A}_\alpha$ of ω with $|\mathfrak{A}_\alpha| = \max\{\omega, |\alpha|\}$ and an infinite subset M_α of N_α. Suppose further that the following hold:

(i) $\mathfrak{A}_\alpha \subset \mathfrak{A}_\beta$ for $\alpha < \beta < \gamma$;

(ii) for all $\alpha < \gamma$ there is a $J_\alpha \subset K_\alpha$ such that $\lambda^{J_\alpha} \in X_{\mathfrak{A}_{\alpha+1}}$;

(iii) for all $\alpha, \beta < \gamma$ we have $M_\alpha \notin \{N_\alpha \cap B : B \in \mathfrak{A}_\beta\}$.

As in Proposition 1E, in the case where γ is a limit ordinal, we put

$\mathfrak{A}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$, and note that $|\mathfrak{A}_\gamma|$ is as required. We have to do some work only when $\gamma = \beta + 1$.

For each $\alpha < \gamma$ we know that M_α is not expressible as $A \cap N_\alpha$ with $A \in \mathfrak{A}_\beta$. Hence, in fact, for each $\alpha < \beta$ and each $A \in \mathfrak{A}_\beta$, $M_\alpha \Delta (A \cap N_\alpha)$ is infinite.

By considering 2^ω almost disjoint infinite subsets of K_β we can show there exists an infinite $J_\beta \subset K_\beta$ such that for no $\alpha < \beta$, $A \in \mathfrak{A}_\beta$ does J_β contain $M_\alpha^\Delta(A \cap N_\alpha)$. Now let $\mathfrak{B}_\beta$ be a countable algebra of subsets of J_β such that $\lambda^J\beta \in X_{\mathfrak{B}_\beta}$. Define $\mathfrak{A}_{\beta+1}$ to be the algebra generated by $\mathfrak{A}_\beta \cup \mathfrak{B}_\beta$.

If B is in $\mathfrak{A}_{\beta+1}$ and $\alpha < \beta$, then $(B \cap N_\alpha)\backslash J_\beta$ is equal to $(A \cap N_\alpha)\backslash J_\beta$ for suitable $A \in \mathfrak{A}_\beta$, and hence cannot equal $M_\alpha \backslash J_\beta$, since J_β did not contain $M_\alpha^\Delta(A \cap N_\alpha)$. Note also that, since $|\mathfrak{A}_\beta| = \max\{\omega, |\beta|\}$ and $|\mathfrak{B}_\beta| = \omega$, we have $|\mathfrak{B}_{\beta+1}| = \max\{\omega, |\beta|\}$ as required. To finish the induction, we choose a subset M_β of N_β which is not of the form $A \cap N_\beta$ with $A \in \mathfrak{A}_{\beta+1}$.

2D Theorem. *There is a compact space T such that $\mathscr{C}(T)$ does not contain l_∞ but does have a bounding subset which is not relatively compact.*

Proof. We take T to be the Stone space of the algebra constructed in Proposition 2C, so that $\mathscr{C}(T)$ can be identified with $X_{\mathfrak{A}}$. The assertions follow from the preceding work and 1C.

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